

Restricted Preference Domains

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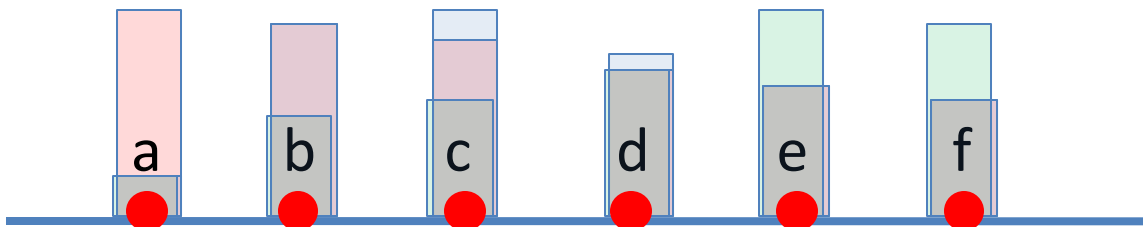


Circumventing Negative Results

- Problem:
with no assumption on preference structure
 - majority cycles may occur
 - all reasonable voting rules are manipulable
 - computing outcomes of some voting rules is NP-hard
- Solution: restrict the preference domain

Single-Peaked Preferences

- Definition: a vote v is **single-peaked (SP)** wrt an ordering $<$ of candidates (axis) if it holds that:
 - if $\text{top}(v) < d < e$, v prefers d to e
 - if $a < b < \text{top}(v)$, v prefers b to a
- Example:
 - voter 1: $c > b > d > e > f > a$
 - voter 2: $a > b > c > d > e > f$
 - voter 3: $e > f > d > c > b > a$



Example: Political Voting

- United Kingdom (specific precinct)
 - candidates: Conservatives (C), Labour (L), Liberal Democrats (LD)
 - 60 000 voters
 - 25 000 voters prefer C to LD to L: $C > LD > L$
 - 20 000 voters: $L > LD > C$
 - 11 000 voters: $LD > L > C$
 - 4 000 voters: $LD > C > L$



Example: Temperature

- Perfect water temperature?

+16

+20

+23

+25

+27

+30

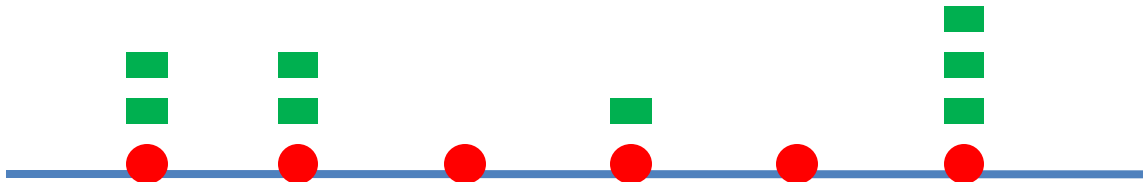


SP Preferences: Transitivity

- Theorem: in single-peaked elections with an odd number of voters the majority relation is transitive
- Corollary: in single-peaked elections with an odd number of voters the Kemeny winner can be computed in polynomial time
- Lemma: each single-peaked election with an odd number of voters has a Condorcet winner (CW))
- Proof of the theorem (assuming the lemma):
 - by the lemma, there is a CW, say a
 - delete a from all votes; the profile remains SP
 - use induction

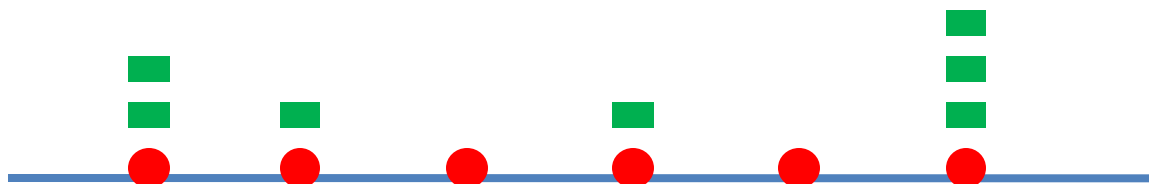
SP Preferences: Condorcet Winners

- Lemma: in single-peaked elections with an odd number of voters there exists a Condorcet winner (CW)
 - ask each voter v to vote for one candidate
 - let $c(v)$ denote the vote of voter v
 - order voters by $c(v)$, breaking ties arbitrarily
 - if we have $n = 2k+1$ voters, $\text{top}(v_{k+1})$ is a CW
 - even n : if we have $n = 2k$ voters, all candidates between $\text{top}(v_k)$ and $\text{top}(v_{k+1})$ are weak CWs



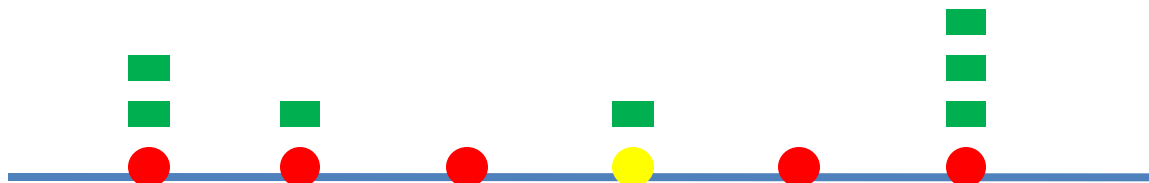
SP Preferences: Non-Manipulability

- Suppose we have $n = 2k+1$ voters
- Median voter rule:
 - consider an election that is single-peaked wrt $<$
 - ask each voter v to vote for one candidate
 - let $c(v)$ denote the vote of voter v
 - order voters by $c(v)$, breaking ties arbitrarily
 - output $c^* = c(v_{k+1})$



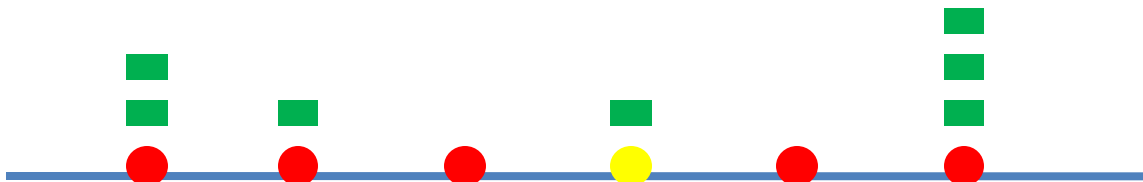
SP Preferences: Median Is Truthful

- Theorem: under the **median voter** rule, no voter can benefit from voting non-truthfully
- Consider a voter v_i in our order
 - $i = k+1$: v_i gets his most preferred outcome
 - $i < k+1$ ($i > k+1$ is symmetric):
 - if v_i votes c , $c \leq c^*$, v_{k+1} remains the median voter, so the outcome **does not change**



SP Preferences: Median is Truthful

- Theorem: under the **median voter** rule, no voter can manipulate
- Consider a voter v_i in our order
 - $i = k+1$: v_i gets his most preferred outcome
 - $i < k+1$ ($i > k+1$ is symmetric):
 - if v_i votes c , $c \leq c^*$, v_{k+1} remains the median voter, so the outcome **does not change**
 - if v_i votes c , $c^* < c$, either v_i (with his new vote) or v_{k+2} becomes the median voter, so the outcome gets **worse** for v_i

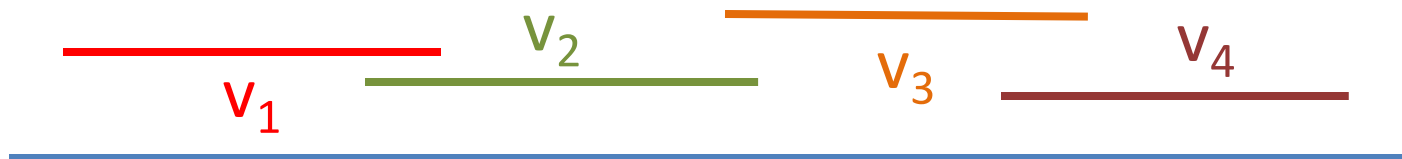


Chamberlin-Courant for Single-Peaked Preferences

- Theorem: if voters preferences are **single-peaked**, we can efficiently compute a committee with maximum **egalitarian/utilitarian** CC score

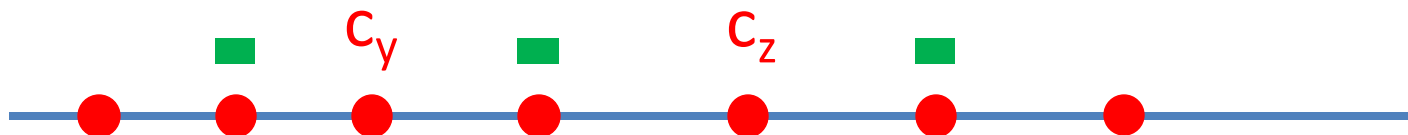
Proof (egalitarian):

- for $s = m - 1, \dots, 1$, we check if there is a committee of size k with egalitarian score $\geq s$
- for each voter, the set of candidates with score s or higher forms a contiguous segment of $<$
- can we pick k points to stab all n intervals? (easy)



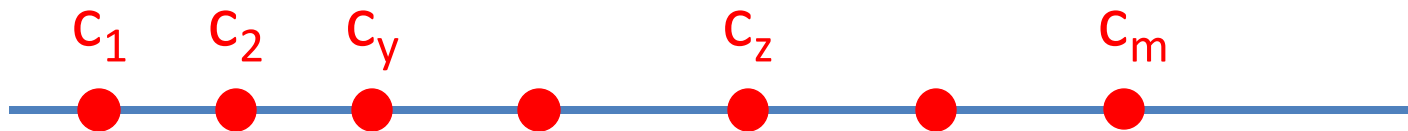
Utilitarian Chamberlin-Courant for Single-Peaked Preferences (Warm-up)

- Marginal contribution of c_z to a committee contained in $\{c_1, \dots, c_y\}$ and containing c_y :
 - suppose $\text{top}(i)$ is to the left of (or equals) c_y
 - then i gains nothing
 - suppose $\text{top}(i)$ is to the right of c_y
 - then i gains $\max\{0, \text{sc}(i, c_z) - \text{sc}(i, c_y)\}$
- Note that we only need to know c_y and c_z to compute the marginal contribution of c_z



Utilitarian Chamberlin-Courant for SP Preferences (Dynamic Program)

- $M(s, z)$: maximum score of a committee that is
 - of size s
 - contained in $\{c_1, \dots, c_z\}$ and contains c_z
- The score of opt committee: $\max_{z=1, \dots, m} M(k, z)$
- For $z = 1, \dots, m$, compute $M(s, z)$ for all $s = 1, \dots, \min\{k, z\}$
- $M(s, z) = \max_{y < z} (M(s-1, y)$
+ marginal contribution of c_z to
a committee contained in $\{c_1, \dots, c_y\}$
and containing c_y)

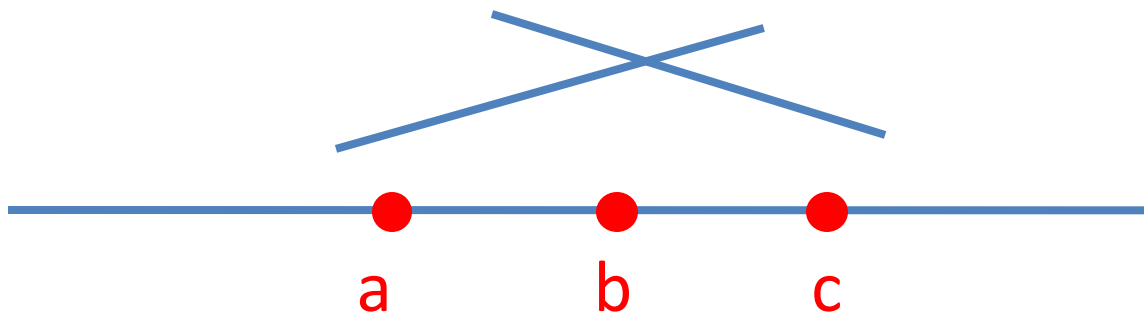


SP Preferences: Equivalent Definitions

- Our definition (decreasing from the peak):
a vote v is SP wrt an ordering $<$ if:
 - if $\text{top}(v) < d < e$, v prefers d to e
 - if $a < b < \text{top}(v)$, v prefers b to a
- Alt definition 1 (no valleys):
a vote v is SP wrt an ordering $<$ if
for every triple of candidates $a < b < c$ it holds that
 b is NOT v 's least preferred candidate in $\{a, b, c\}$
- Alt definition 2 (contiguous segments):
a vote v is SP wrt an ordering $<$
if for every k the set of top k candidates in v
forms a contiguous segment wrt $<$

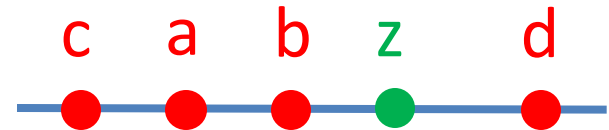
Equivalent Definitions: Proofs

- decreasing from peak $\Rightarrow$ no valleys
 - suppose for contradiction v ranks b last of $\{a, b, c\}$
 - suppose $\text{top}(v)$ is to the left of b
 - then we have $\text{top}(v) < b < c$, so v must prefer b to c
 - suppose $\text{top}(v)$ is to the right of b
 - then we have $a < b < \text{top}(v)$, so v must prefer b to a



Equivalent Definitions: Proofs

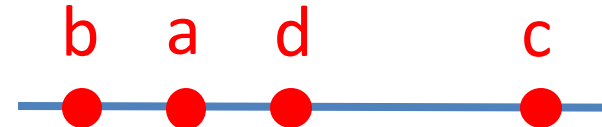
- no valleys $\Rightarrow$ contiguous segments
 - suppose for contradiction the set of top k candidates in v is not contiguous
 - pick the smallest such k
 - let d be v 's k -th candidate
 - let z be a candidate that separates d on the axis from the top $k-1$ candidates in v
 - assume wlog $\text{top}(v) < z < d$
 - v ranks z below d , so $\{\text{top}(v), z, d\}$ forms a valley



a
b
c
d
.
.
z
.

Equivalent Definitions: Proofs

- contiguous segments $\Rightarrow$ decreasing from peak
 - suppose for contradiction $\text{top}(v) < d < c$, yet v prefers c to d
 - suppose v ranks c in position k
 - then v ranks d in position $k+1$ or lower
 - thus the set of $\text{top } k$ candidates in v is not contiguous wrt $<$

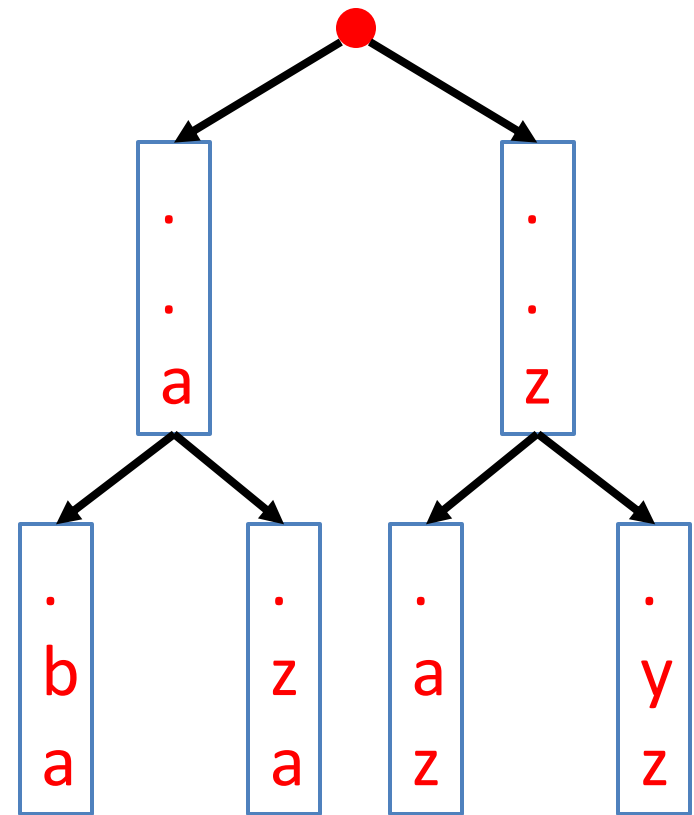
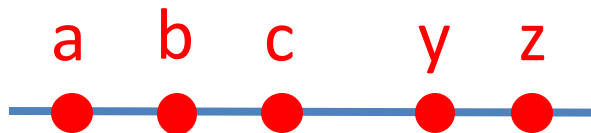


SP Preferences: Properties

- Claim: if a profile is single-peaked, then the set of candidates that are ranked last by at least one voter has size ≤ 2
 - only the endpoints of the axis can be ranked last
- Claim: if $a < b < \dots < z$ is an axis for a profile P , then so is $z < \dots < b < a$
- Claim: if a single-peaked profile contains votes $a \succ b \succ \dots \succ z$ and $z \succ \dots \succ b \succ a$, then the only axes for it are $a < b < \dots < z$ and $z < \dots < b < a$
 - a and z have to be the endpoints

SP Preferences: Properties

- Claim: for a given axis $<$ over m candidates, there are 2^{m-1} different votes that are SP wrt $<$
 - build up a vote SP wrt $<$ from the bottom up
 - 2 options at each but last step

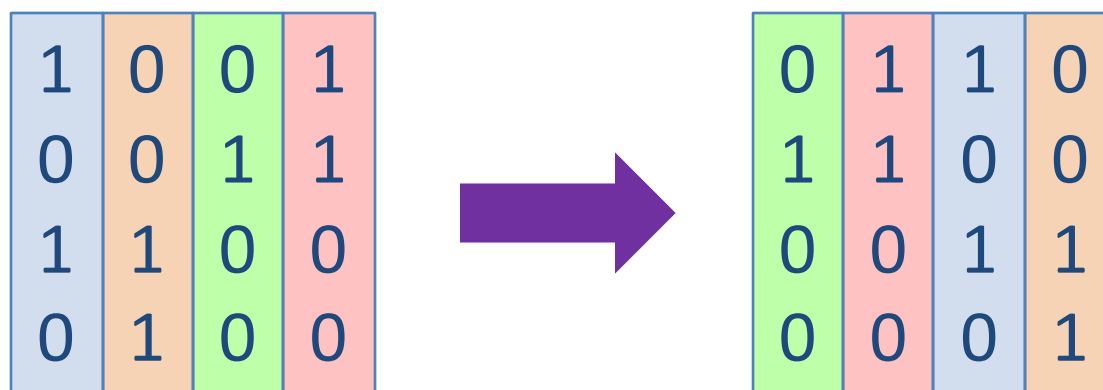


Recognizing SP Preferences

- It is easy to check if a profile is single-peaked wrt a **given axis**
- But can we efficiently check if a profile is single-peaked wrt **some axis**?
- Turns out that the answer is “yes”
 - via a powerful tool
 - via a direct combinatorial linear-time algorithm

Consecutive 1s

- Consecutive 1s problem: given a 0-1 matrix M , can we reorder the columns of M so that 1s appear *consecutively* in each row?
 - admits a polynomial-time algorithm (PQ-trees)



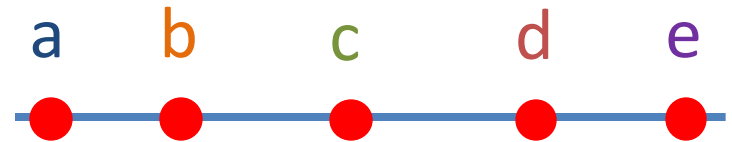
Recognizing SP Preferences

- $v_1: b \succ a \succ c \succ d \succ e$
- $v_2: c \succ d \succ b \succ e \succ a$
- $v_3: d \succ e \succ c \succ b \succ a$
- if we can arrange the columns so that all 1s are consecutive, then each prefix is contiguous

a	b	c	d	e
1	1	0	0	0
1	1	1	0	0
1	1	1	1	0

Recognizing SP Preferences Combinatorially (sketch)

- $v_1: b \succ a \succ c \succ d \succ e$
- $v_2: c \succ d \succ b \succ e \succ a$
- $v_3: d \succ e \succ c \succ b \succ a$



- Alternatives ranked last by some voter: $\{a, e\}$
- $v_1: b \succ c \succ d$
- $v_2: c \succ d \succ b$
- $v_3: d \succ c \succ b$
- Alternatives ranked last by some voter: $\{b, d\}$
- $v_3: e \succ b \succ a$, $\text{top}(v_3) \neq b$ and not yet placed
- Hence b has to be next to a
 - if b is next to e , then $\text{top}(v_3)$, b and e form a valley

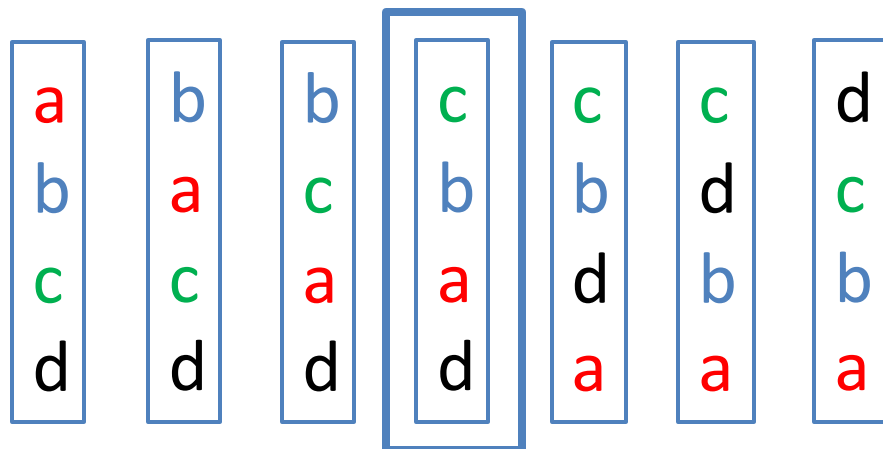
Single-Crossing Preferences

Definition: a profile is **single-crossing (SC)** wrt an ordering of voters $(v_1, \dots, v_n)$ if for each pair of candidates **a, b** there exists an $i \in \{0, \dots, n\}$ such that voters $v_1, \dots, v_i$ prefer **a** to **b**, and voters $v_{i+1}, \dots, v_n$ prefer **b** to **a**

a	b	b	c	c	c	d
b	a	c	b	b	d	c
c	c	a	a	d	b	b
d	d	d	d	a	a	a

SC Preferences: Desirable Properties

- Claim: in single-crossing elections, the majority relation is (weakly) transitive
 - we will prove the claim for $n=2k+1$ voters
 - consider the ranking of voter v_{k+1}
 - if v_{k+1} prefers b to a , so do $\geq k$ other voters
- Claim: in single-crossing elections, Chamberlin-Courant winners can be computed in poly-time by dynamic programming

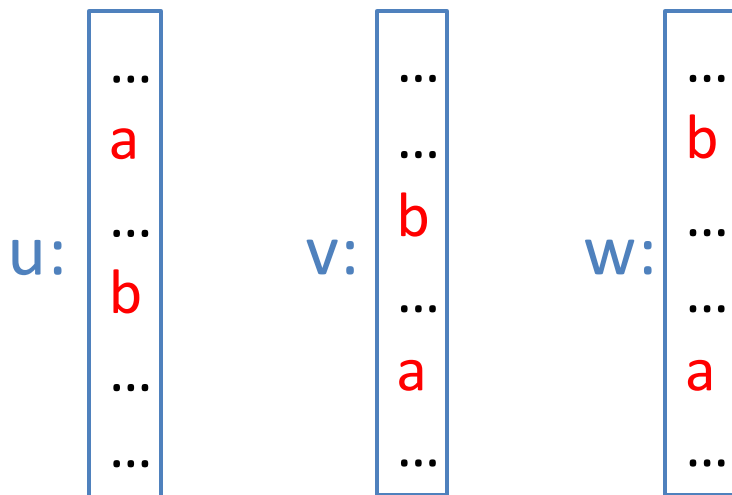


Recognizing SC Preferences

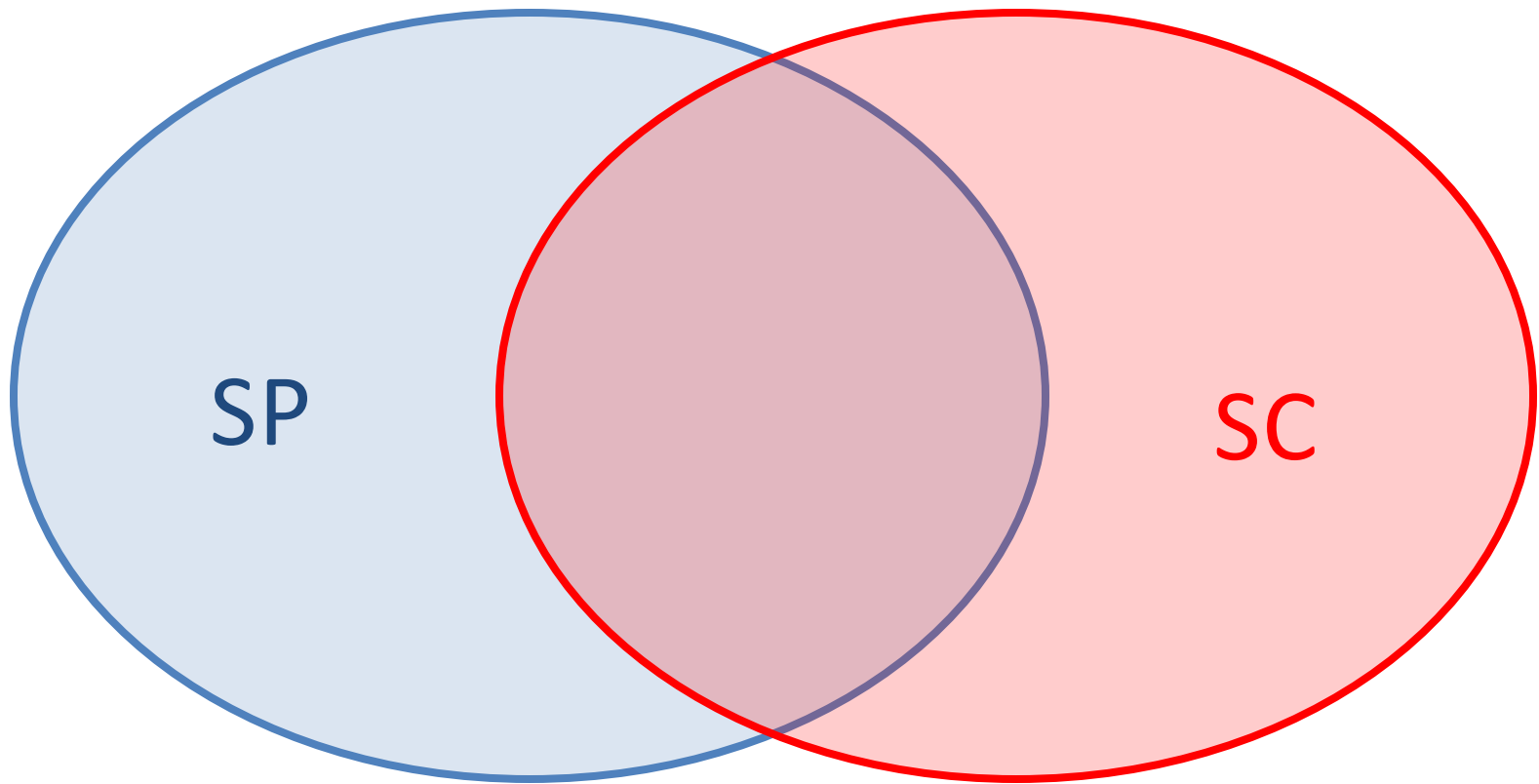
- Reduction to consecutive 1s
- a **column** for each **voter**
- a **row** for each **ordered pair** of candidates
- In the **(a, b)** row,
we have 1s for
all voters who prefer **a** to **b**

Recognizing SC Preferences, Combinatorially

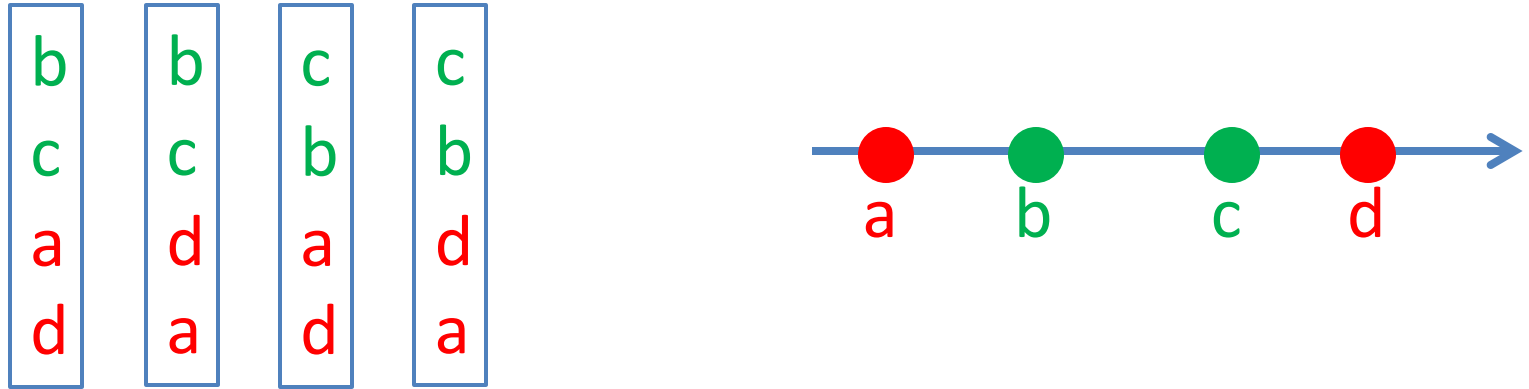
- Assume wlog all votes are distinct
- $d_{\text{swap}}(x, y): |\{(a, b): x \text{ prefers } a \text{ to } b, y \text{ prefers } b \text{ to } a\}|$
- Lemma: if $u < v < w$, then $d_{\text{swap}}(u, v) < d_{\text{swap}}(u, w)$
- Theorem: for each vote u , can decide in **poly-time** if there is a **SC** ordering where u appears first
- Try all possibilities for the first vote



???



Single-Peaked Profile That Is Not Single-Crossing



- v_1 and v_2 have to be adjacent (because of b, c)
- v_3 and v_4 have to be adjacent (because of b, c)
- v_1 and v_3 have to be adjacent (because of a, d)
- v_2 and v_4 have to be adjacent (because of a, d)

a contradiction

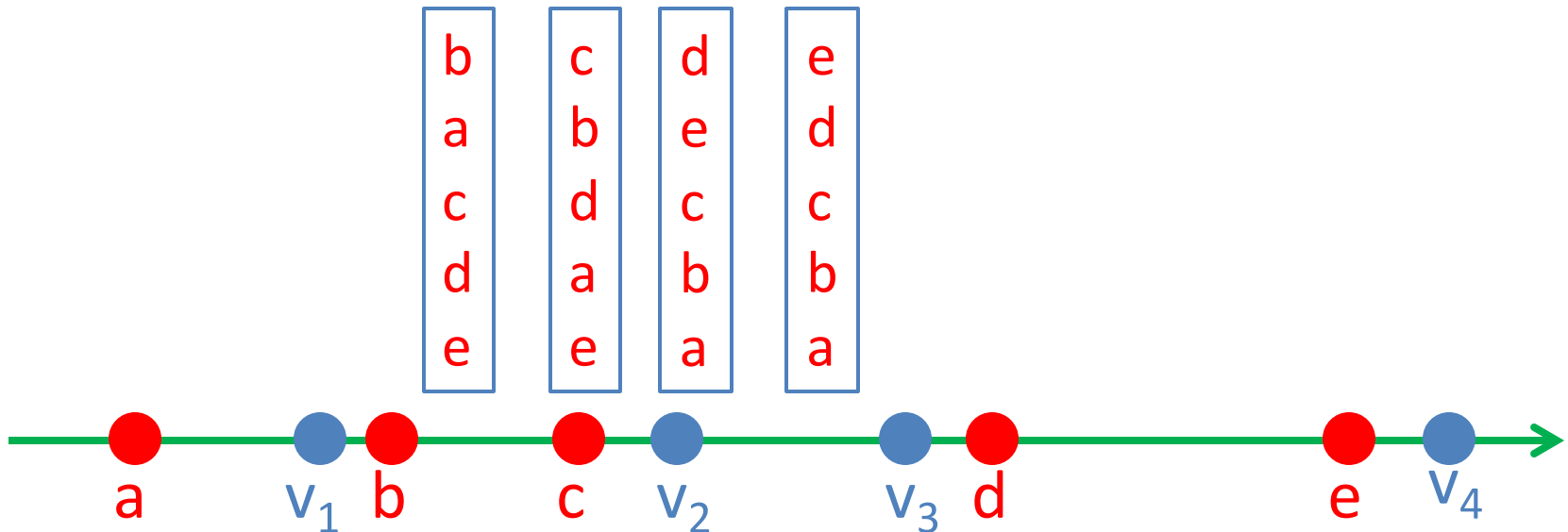
Single-Crossing Profile That Is Not Single-Peaked

1	n	n		n	n
2	1	n-1		n-1	n-1
...	2	1		n-2	n-2
...	...	2		...	...
...	...	...	...	...	...
n-2	...	...		...	...
n-1	n-2	...		1	2
n	n-1	n-2		2	1

Each candidate is ranked last **exactly once**

1D-Euclidean Preferences

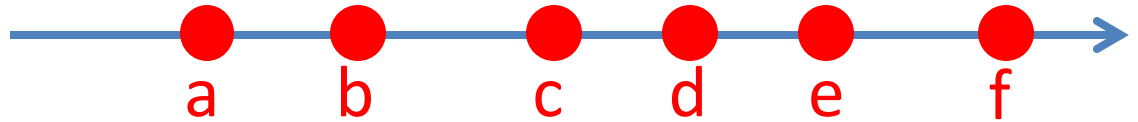
- Both **voters** and **candidates** are points in $\mathbb{R}$
- v prefers a to b if $|v - a| < |v - b|$
- Observation: 1D-Euclidean preferences are
 - **single-peaked** (wrt ordering of **candidates** on the line)
 - **single-crossing** (wrt ordering of **voters** on the line)



1-Euc = SP $\cap$ SC?

- Observation: There exists a preference profile that is **SP** and **SC**, but not 1-Euclidean

v_1 : **b c d e a f**
 v_2 : **d e c b a f**
 v_3 : **d e f c b a**



- **SC** wrt $v_1 < v_2 < v_3$, **SP** wrt $a < b < c < d < e < f$
- Not 1-Euclidean:
 - $(x(a) + x(e))/2 < x(v_1) < (x(b) + x(c))/2$
 - $(x(c) + x(d))/2 < x(v_2) < (x(a) + x(f))/2$
 - $(x(b) + x(f))/2 < x(v_3) < (x(d) + x(e))/2$

